Power Series

We fix a $k$-Banach space where $k$ is a nonarchimedean field.

Definition: By a formal power series in $r$ variables, with coefficients in $V$, we mean a formal series:

$$f(X) = \sum_{\alpha} X^\alpha V_{\alpha}, \quad V_{\alpha} \in V.$$ 

Here $\alpha$ runs in $(\mathbb{N} \cup \{0\})^r$ and if $\alpha = (\alpha_1, \ldots, \alpha_r)$ then

$$|\alpha| = \alpha_1 + \ldots + \alpha_r$$

and

$$X^\alpha = X_1^{\alpha_1} \ldots X_r^{\alpha_r}.$$ 

Given $\varepsilon > 0$, we say $f(X)$ is $\varepsilon$-convergent if

$$\lim_{k \to \infty} \varepsilon^{k^1} \|V_{\alpha_k}\| = 0.$$ 

Notice that if $f(X)$ is $\varepsilon$-convergent and $0 \leq \delta \leq \varepsilon$ then

$$0 \leq \delta^{k^1} \|V_{\alpha_k}\| < \varepsilon^{k^1} \|V_{\alpha_k}\| \to 0$$

so $f(X)$ is also $\delta$-convergent.
Fact: Denote, for a given \( \varepsilon > 0 \), by \( \mathcal{F}_\varepsilon(k; V) \) the space of all \( \varepsilon \)-convergent power series. We have, then, that \( \mathcal{F}_\varepsilon(k; V) \) is a Banach Space with the norm

\[
\|f\|_\varepsilon := \max_{x \in k} \varepsilon^{|x|} \|V^x\|_\text{all}.
\]

We will not prove this fact but we will always understand this is the norm it's used.

Our aim in this lecture is to prove, or state only in some cases, the major results of power series that we will need in our study of manifolds.

We begin with the following:

Proposition: The following properties are true for \( \mathcal{F}_\varepsilon(k; V) \):

1. \((V \text{-norm independence})\): The topology of \( \mathcal{F}_\varepsilon(k; V) \) does not depend on the norm of \( V \), only on the topology it gives to it.

2. \((\text{Coefficient wise limits})\): Suppose we have a convergent series

\[
f = \sum_{i=1}^{\infty} f_i
\]
in $F_e(k^r; V)$, say,

$$f(x) = \sum_\alpha x^\alpha v_\alpha$$

and

$$f_2(x) = \sum_\alpha x^\alpha v_{\alpha, i}.$$ 

Then we have

$$v_\alpha = \sum_{i=1}^{\infty} v_{\alpha, i}.$$ 

3. (Cauchy Product): Let $u : V_1 \times V_2 \to V$ be a continuous bilinear mapping between $k$-Banach spaces; then

$$U : F_e(k^r; V_1) \times F_e(k^r; V_2) \to F_e(k^r; V)$$

defined by

$$u\left(\sum_\alpha x^\alpha v_\alpha, \sum_\alpha x^\alpha w_\alpha\right) \mapsto \sum_\alpha x^\alpha \sum_{\beta \odot \gamma = \alpha} u(v_\beta, w_\gamma)$$

is a continuous bilinear mapping.

4. (Banach Algebra): $F_e(k^r; k)$ is a commutative $k$-algebra with multiplication given by the Cauchy product, moreover we have

$$\|f + g\|_\varepsilon = \|f\|_\varepsilon + \|g\|_\varepsilon.$$
5. Composition: Let \( g \in \mathbb{F}_\varepsilon(k^r; k^n) \) such that 
\[ \|g\| \leq \varepsilon, \]
then
\[
\mathbb{F}_\varepsilon(k^r; V) \xrightarrow{g} \mathbb{F}_\varepsilon(k^r; V)
\]

\[
\sum'_\beta Y^B_v \beta \xrightarrow{g} \sum'_\beta g(x)^B_v \beta = fog(x)
\]
is a continuous linear map of norm less or equal to 1.

Proof:
1. Let \( \| \cdot \| \) and \( \| \cdot \|' \) be the two norms in \( V \). We can find constants \( c_1, d, c_2 \) relating the norms by

\[
c_1 \| \cdot \| \leq \| \cdot \|' \leq c_2 \| \cdot \|
\]

Hence, given a power series \( f(x) = \sum V_a x^a \), we have

\[
\varepsilon^{k_1} \| V_a \| \leq \frac{1}{c_1} \varepsilon^{k_1} \| V_a \|
\]

and

\[
\varepsilon^{k_1} \| V_a \|' \leq c_2 \varepsilon^{k_1} \| V_a \|
\]

Here, if \( f(x) \) is in \( \mathbb{F}_\varepsilon(k^r; V) \) with one topology, it is there with the other one. Hence the space consist of exactly the same power series.

Moreover we have
$$\|f\| = \max_x \epsilon^{\|x\|} v_{\epsilon x}$$

$$\leq \max \frac{1}{\epsilon} \epsilon^{\|x\|} v_{\epsilon x}$$

$$= \frac{1}{\epsilon} \max \epsilon^{\|x\|} v_{\epsilon x}$$

$$= \frac{1}{\epsilon} \|f\|$$

and analogously \( \|f\| \leq C_2 \|f\| \). That is,

\[ c_1 \|f\| \leq \|f\| \leq c_2 \|f\|, \]

so that the norms are \emph{equivalent}, i.e., we have the same topology.

2. We have

\[
f - \sum_{i=1}^{n} f_i = \sum_{x} x^x v_{x} - \sum_{i=1}^{n} x^x v_{x_i, i}
\]

\[
= \sum_{x} x^x (v_{x} - v_{x_i, i})
\]

since it is a finite sum. Taking norm:

\[
\|f - \sum_{i=1}^{n} f_i\| = \| \sum_{x} x^x (v_{x} - v_{x_i, i}) \| = \max \epsilon^{\|x\|} \| v_{x} - v_{x_i, i} \|
\]

Taking a limit we have
\[ \lim_{\varepsilon \to 0} \max_{\alpha} \| V_{\alpha} - V_{\alpha,0} \| = 0, \]

which implies, all terms for a fixed \( \alpha \),

\[ \lim_{\varepsilon \to 0} \| V_{\alpha,0} - V_{\alpha,0,1} \| = 0. \]

To see why, pick \( \varepsilon_0 > 0 \) arbitrary, and \( N > 0 \) such that \( n \geq N \) implies

\[ \max_{\alpha} \varepsilon_{\mathrm{hol}} \| V_{\alpha} - V_{\alpha,0,1} \| = 0. \]

Then we have for \( n \geq N \),

\[ \varepsilon_{\mathrm{hol}} \| V_{\alpha,0} - V_{\alpha,n} \| \leq \max_{\alpha} \varepsilon_{\mathrm{hol}} \| V_{\alpha} - V_{\alpha,0,1} \| < \varepsilon_0. \]

i.e.

\[ \| V_{\alpha,0} - V_{\alpha,0,n} \| < \varepsilon_0, \]

as desired.

\[ \therefore \ V_{\alpha,0} \longrightarrow V_{\alpha}. \]

3.- A bilinear mapping \( B \) between Banach Spaces is continuous iff

\[ \| B(V_1, V_2) \| \leq c \| V_1 \| \| V_2 \|, \]

for some constant \( c > 0 \). This is what we will prove. That the map \( E \times D \) bilinear is obvious.
We already know \( u \) is continuous so such a constant \( c > 0 \) exists for it. Hence:

\[
\varepsilon^{k_1} \left\| \sum_{\beta + \alpha = \alpha} u(v_\beta, W_\alpha) \right\| \leq \varepsilon^{k_1} \max_{\beta + \alpha = \alpha} \left\| u(v_\beta, W_\alpha) \right\|
\]

by strict \( \Delta \) inequality

\[
\leq \varepsilon^{k_1} \max_{\beta + \alpha = \alpha} c \left\| v_\beta \right\| \left\| W_\alpha \right\|
\]

\[
= c \varepsilon^{k_1} \max_{\beta + \alpha = \alpha} \left\| v_\beta \right\| \left\| W_\alpha \right\|
\]

since \( \varepsilon^{k_1} \) is a constant with respect to the variables \( \beta, \alpha \).

\[
= c \varepsilon^{k_1} \max \left( \varepsilon^{1/k_1} \left\| v_\beta \right\| \left\| W_\alpha \right\| \right)
\]

since \( \| x \| = |\beta| + |\alpha| \)

\[
\leq c \max_{\beta} \left( \varepsilon^{k_1} \left\| v_\beta \right\| \right) \cdot \max_{\beta} \left( \varepsilon^{1/k_1} \left\| W_\beta \right\| \right)
\]

Hence, taking maximum and noticing the RHS does not depend on \( \alpha \), we have:

\[
\max_{\alpha} \varepsilon^{k_1} \left\| \sum_{\beta + \alpha = \alpha} u(v_\beta, W_\alpha) \right\| \leq c \max_{\beta} \left( \varepsilon^{k_1} \left\| v_\beta \right\| \right) \cdot \max_{\beta} \left( \varepsilon^{1/k_1} \left\| W_\beta \right\| \right)
\]

that is,

\[
\left\| f \right\| \left\| u(f, g) \right\| \leq C \left\| f \right\| \left\| g \right\|,
\]

as desired.
4. All follow by \( (3) \) except

\[ ||f|| = ||f|| \cdot ||g||. \]

(3) only proves \( ||f|| \leq ||f|| \cdot ||g|| \), since for multiplication the constant \( c \) can be taken to be 1. We prove now

\[ ||f|| \geq ||f|| \cdot ||g||. \]

We put \( N_z = (N_{u \cdot f} s^1) \) the lexicographic order and define \( \mu + \nu \) to be minimal such that

\[ \#{\mu} \leq l_{\mu} e^1, \quad \#{\nu} \leq l_{\nu} e^1, \quad \] and define \( \lambda = \beta + \delta \cdot \mu + \nu \). \( \lambda \) will work as our pivot to compare: the maximum \( ||f|| \) should be bigger to what happens in level \( \lambda \), and, by our definition, in this level we can recover exactly the norm \( ||f|| \cdot ||g|| \).

**Claim:** \( \beta \leq \mu \) or \( \delta \leq \nu \) if \( \beta + \delta \cdot \mu + \nu \).

**Proof:** Otherwise we have \( \beta > \mu \) and \( \delta > \nu \).

That is

\[ \beta = \mu, \quad \delta_i = \mu_i - 1, \quad \beta_i > \mu_i, \]

and

\[ \delta_i = \nu_i, \quad \delta_i = \nu_i - 1, \quad \delta_i > \nu_i. \]
Suppose that \( i < j \), then

\[
\lambda_i = \mu_i + \nu_i < \beta_i + \xi_i = \lambda_i,
\]

and this is a contradiction. \(\Box\).

For any \( \beta + \delta = \lambda \) we have, by definition of \( \| \cdot \| \varepsilon \):

\[
|\beta| \epsilon^{181} \leq \| f \| \varepsilon = |b_{\mu}| \epsilon^{181}.
\]

\[
|c_{\delta}| \epsilon^{181} \leq \| g \| \varepsilon = |c_{\nu}| \epsilon^{181}.
\]

and by the claim, if \( (\beta, \delta) \neq (\mu, \nu) \) then \( \beta < \mu \) or \( \delta < \nu \), hence one of the previous inequalities is strict. We get

\[
|b_{\beta} c_{\delta}| \epsilon^{181} = |b_{\beta}| \epsilon^{181} \cdot |c_{\delta}| \epsilon^{181} < \| f \| \varepsilon \| g \| \varepsilon,
\]

whenever \( (\beta, \delta) = (\mu, \nu) \) and \( (\beta, \delta) \neq (\mu, \nu) \). We conclude:

\[
\| f \| \varepsilon \geq \left| \sum_{\beta + \delta = \lambda} b_{\beta} c_{\delta} \right| \epsilon^{181}
\]

\[
= \max_{\beta} \{ |b_{\mu} c_{\nu}| \epsilon^{181} \}
\]

\[
= \| f \| \varepsilon \| g \| \varepsilon,
\]

as desired.
Notice that we have

\[ f_\otimes (K^n, K^n) = \underbrace{f_\otimes (K^r, K) \times \cdots \times f_\otimes (K^r, K)}_{n \text{ times}} \]  

Write, using this identification, \( g = (g_1, \ldots, g_n) \). Then

\[ g(Y)^\beta = g_1(Y)^{\beta_1} \cdots g_n(Y)^{\beta_n}. \]

For a fixed \( \beta = (\beta_1, \ldots, \beta_n) \), then we can consider \( g(\cdot) \) we have \( g \in f_\otimes (K^r, K) \) and so on.

We want to prove \( g(X)^\beta \in f_\otimes (K^r, K) \): For this we know that

1. \( g \in f_\otimes (K^r, K) \Rightarrow g_\beta^\gamma \in f_\otimes (K^r, K) \) by proposition part (4).

2. \( g_1^\beta_1 \cdot g_2^\beta_2 \cdots g_n^\beta_n \in f_\otimes (K^r, K) \) by part (4) again.

Moreover, the norm equality gives then:

\[ \|g_1^\beta_1 \cdots g_n^\beta_n\| = \|g_1^\beta_1\| \cdots \|g_n^\beta_n\| = \|g_1\|^{\beta_1} \cdots \|g_n\|^{\beta_n}. \]

Now notice that the reduced \( K \)-Banach space.
\[ \| \sum_{\beta} X^\beta (k_{\beta}^{(1)}, \ldots, k_{\beta}^{(n)}) \|_\beta = \max_i \delta^{i\beta} \| (k_{\beta}^{(1)}, \ldots, k_{\beta}^{(n)}) \|_\beta \]

\[ = \max_i \delta^{i\beta} \max_j (1k_{\beta}^{(1)}, \ldots, 1k_{\beta}^{(n)}) \]

\[ = \max_i \delta^{i\beta} \max_j (0k_{\beta}^{(1)}, \ldots, 0k_{\beta}^{(n)}) \]

\[ = \max_i \delta^{i\beta} \max_j (1) \]

\[ = \max_i \| g_i \|_\beta \]

\[ = \| (g_1, \ldots, g_n) \|. \]

Hence \((x)\) is an isometry and we get, for the case of \(g^\beta\) that:

\[ \| g^\beta \| = \| g_1 \|, \ldots, \| g_n \| = \| g_1 \|, \ldots, \| g_n \| \leq \varepsilon^{\beta 1} \]

Identification as an isometry.

Hence we have \(g^\beta \in F_k (k^r; k)\) and hence \(g^\beta (x) v_\beta \in F_k (k^r; V)\) since

\[ \max_\beta \| g^\beta (x) v_\beta \| \leq \max_\beta \varepsilon^{\beta 1} \| v_\beta \| \rightarrow 0 \]

since \(f \in F_k (k^r; V)\). Even more is true, this also implies that...
\[ \lim_{n \to \infty} \| g^n(x) v_n \| = 0 \] and so the series
\[ \sum_{n} g^n(x) v_n \]
converges, since its general term goes to 0, in \( F^k(K; V) \). Finally:
\[ \| f g \|_\sigma = \max_{n \geq 1} \| g(x)^n v_n \|_\sigma \]
\[ = \max_{n \geq 1} \| g(x)^n \| \| v_n \| \]
\[ = \| f \| \| g \| \]
giving that it has norm less than or equal to 1.

We of course have evaluation in the right neighbourhood, which explains why the \( \epsilon > 0 \) in the previous definitions.

**Definition:** Define for a \( \{z^n \} \) power series \( f \in F_e(k^n; V) \) an evaluation map
\[ e : F_e(k^n; V) \to \text{Maps}(B_e(0), V) \]
\[ f \mapsto \tilde{f} : B_e(0) \to V \]
\[ x \mapsto \sum_{\beta} x^\beta v_\beta. \]
This is well defined since \( x \in \mathbf{B}_e(0) \) implies

\[
\| \sum_{\alpha} x^\alpha v_\alpha \| = \max_{\alpha} \| x^\alpha v_\alpha \| \leq \| \mathbf{e}^x V_{\text{all}}\| = \| f \| _e.
\]

Properties of evaluation: We have:

1. With \( U \) defined as in (3) of previous proposition

\[
U(f, g)(x) = u(\tilde{f}(x), \tilde{g}(x))
\]

for \( x \in \mathbf{B}_e(0) \).

2. For the case of multiplication we have

\[
(fg)(x) = \tilde{f}(x) \tilde{g}(x), \quad x \in \mathbf{B}_e(0).
\]

3. For composition

\[
(f \circ g)(x) = \tilde{f}(\tilde{g}(x)) \quad x \in \mathbf{B}_e(0) \subseteq A.
\]

We do not prove this one, it follows from the previous one using its properties now at \( V \), where the results are known. The following will be a fundamental type of result for \( w \):
Point of Expansion: Let $f \in \mathcal{F}_e(k', V)$ and $y \in B_\varepsilon(0)$; then there exists an $f_y \in \mathcal{F}_e(k', V)$ such that

$$
\|f_y\|_e = \|f\|_e
$$

and

$$
\tilde{f}(x) = f_y(x - y) \quad \forall x \in B_\varepsilon(0) = B_\varepsilon(y).
$$

Proof:

Let $e_1, \ldots, e_r$ be the basis of $k'$. Consider the power series

$$
g(x) = x + y = \sum_{i=1}^r x_i e_i + y,
$$

where $e_i \in \mathcal{F}_e(k', k')$, i.e., $g$ satisfies $\|g\|_e \leq \varepsilon$. Hence, we can apply part (3) of previous proposition and part (5) of previous one to get

$$
f_y(x) = f(x + y) \in \mathcal{F}_e(k', V)
$$

with

$$
\|f_y\|_e \leq \|f\|_e,
$$

$$
\tilde{f}(x) = f(x - y) \quad \forall x \in B_\varepsilon(0).
$$

Finally, by symmetry, we have:

$$
\|f \|_e = \|f(y) - y\|_e \leq \|f_y\|_e.
$$
This idea leads to:

**Definition:** A function $f: U \rightarrow V$, with $U \subseteq \mathbb{R}^r$ an open set, is called a locally analytic function if for any point $x_0 \in U$ there exists a ball $B_{e}(x_0) \subseteq U$ around $x_0$ and a power series $F \in \mathcal{F}(r; V)$ such that

$$f(x) = F(x - x_0) \quad \forall x \in B_{e}(x_0)$$

(We are denoting $F$ and its evaluation at $x-x_0$ by the same notation.)

We denote the set of locally analytic functions $f: U \rightarrow V$ by $\mathcal{C}^{an}(U, V)$.

**Proposition:** $\mathcal{C}^{an}(U, V)$ is a $\mathbb{K}$-vector space with respect to pointwise addition. Moreover, for any $F \in \mathcal{F}(r; V)$ we have $\tilde{F} \in \mathcal{C}^{an}(B_{e}(0), V)$.

**Proof:**

1) Let $F \in \mathcal{F}(r; V)$ such that

$$f_i(x) = F_i(x - x_0) \quad \forall x \in B_{e}(x_0).$$

Picking $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ we get that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$= F_1(x - x_0) + F_2(x - x_0) \quad \forall x \in B_{e}(x_0)$$

$$= (F_1 + F_2)(x - x_0) \quad \forall x \in B_{e}(x_0).$$
(iii) Finally, the last part is exactly point of expansion.

Properties of locally analytic functions:

1. For any open set \( U' \subseteq U \) we have the linear restriction map

\[
C^\infty(U, V) \longrightarrow C^\infty(U', V)
\]

\[
f \longmapsto f|_{U'}. 
\]

2. For any open and closed set \( U' \subseteq U \) we have the linear map

\[
C^\infty(U', V) \longrightarrow C^\infty(U, V)
\]

\[
f \longmapsto f_1
\]

given by

\[
f_1(x) = \begin{cases} f(x) & x \in U' \\ 0 & \text{otherwise} \end{cases}
\]

It's called extension by zero.

3. If \( U = \bigsqcup_i U_i \) is a covering by pairwise disjoint open subsets then

\[
C^\infty(U, V) \cong \prod_i C^\infty(U_i, V)
\]

\[
f \longmapsto \left( f_i \right)_{i \in I}
\]
4.- For any two $k$-Banach Spaces $V$ and $W$ we have

\[ C^\infty(U, V \oplus W) \cong C^\infty(U, V) \oplus C^\infty(U \oplus W) \]

\[ f \mapsto (\text{pr}_V f, \text{pr}_W f) \]

In particular

\[ C^\infty(U, k^n) \cong \prod_{i=1}^n C^\infty(U, k). \]

5.- For any continuous bilinear map $u : V_1 \times V_2 \to V$ between $k$-Banach Spaces we have the bilinear map

\[ C^\infty(U, V_1) \times C^\infty(U, V_2) \to C^\infty(U, V) \]

\[ (f, g) \mapsto u(fg). \]

In particular, $C^\infty(U, k)$ is a $k$-algebra and $C^\infty(U, V)$ is a module over $C^\infty(U, k)$.

6.- For any continuous linear map $u : V \to W$ between $k$-Banach Spaces we have the linear map

\[ C^\infty(U, V) \to C^\infty(U, W) \]

\[ f \mapsto u \circ f \]
Proposition: Let $U' \subseteq k^n$ be an open subset and let $g \in C^0(U, k^n)$ such that $g(U) \subseteq U'$; then the map

\[ C^0(U', V) \rightarrow C^0(U, V) \]

\[ f \rightarrow f \circ g \]

is well defined and $k$-linear.

Proof:
Let $x_0 \in U$ and put $y_0 := g(x_0) \in U'$.

Choose a ball $B_{e}(y_0) \subseteq U'$ and a power series $F \in \mathcal{F}(k^n, V)$ such that

\[ f(y) = F(y - y_0) \quad \forall y \in B_{e}(y_0). \]

Pick a ball now $B_{e}(x_0) \subseteq U$ and a power series $G \in \mathcal{F}(k^n, k^n)$ such that

\[ g(x) = G(x - x_0) \quad \forall x \in B_{e}(x_0). \]

Now notice that: If $G(x) = \sum_{\alpha} x^\alpha v_\alpha$ then
for \( 0 < \sigma' < \sigma \) we have:

\[ \| G - G(0) \|_\sigma = \left\| \sum_{k \neq 0} x^k V_{ax} \right\|_{\| \cdot \|_{\sigma'}} = \max_{|k| \neq 0} |\sigma'|^{|k|} \left\| V_{ax} \right\|_{\| \cdot \|_{\sigma'}} \]

\[ = \max_{|k| \neq 0} (|\sigma'|)^{|k|} \sigma^{-|k|} \left\| V_{ax} \right\|_{\| \cdot \|_{\sigma'}} \]

\[ \geq \max_{|k| \neq 0} (\frac{\sigma'}{\sigma})^{|k|} \sigma^{-|k|} \left\| V_{ax} \right\|_{\| \cdot \|_{\sigma'}} \text{ since } |k| \geq 1 \]

\[ = \frac{\sigma'}{\sigma} \max_{|k| \neq 0} (\sigma')^{|k|} \left\| V_{ax} \right\|_{\| \cdot \|_{\sigma'}} \]

\[ = \frac{\sigma'}{\sigma} \| G(0) \|_{\| \cdot \|_{\sigma'}} \]

\[ \longrightarrow 0 \quad \text{as} \quad \sigma' \longrightarrow 0. \]

Hence we suppose we have picked a \( \sigma > 0 \) such that

\[ \| G(0) \|_{\| \cdot \|_{\sigma'}} < \varepsilon \]

Notice since we do this by decreasing \( \sigma' \) to \( 0 \), we still belong to \( \mathcal{F}_{\sigma'}(R_{k'j'k''}) \).
Notice that \( G(0) = y_0 \) and that this inequality implies:

\[
x \in B_\varepsilon(x_0) \implies g(x) = G(x - x_0)
\]

\[
= \| g(x) - y_0 \| \leq \| G(x - x_0) - G(0) \|
\]

\[
\leq \max_{x \in B_\varepsilon(x_0), \| x - 0 \| = \varepsilon} \| G(x - x_0) - G(0) \|
\]

\[
\leq \| G(x) - G(0) \|
\]

\[
\leq \varepsilon
\]

we conclude

\[
g(B_\varepsilon(x_0)) \subseteq B_\varepsilon(y_0)
\]

Hence we can compose the power series and get

\[
F_0(G - y_0) \in F_0(k^r, V) \quad \text{and} \quad [F_0(G - y_0)](x - x_0) = F(G(x - x_0) - y_0)
\]

\[
= F(g(x) - y_0)
\]

\[
= f(g(x)) \quad \text{as desired.}
\]