Compact $p$-adic Lie Groups.

In the following we will assume $G$ is a locally analytic manifold over $\mathbb{Q}_p$. Our aim is to prove and discuss the following theorem:

**Theorem 27.1:** Let $G$ be any $p$-adic Lie group; then there exist a compact open subgroup $G' \subseteq G$ and an integral valued $p$-valuation $\omega$ on $G'$ defining the topology of $G'$ such that we have:

i) $(G', \omega)$ is saturated;

ii) $\text{rank } G' = \dim G$.

We will divide this into several steps:

**Step 1:** Building up the chart.

Let $d = \dim G'$. We denote the identity of $G$ by $e$ and we pick a chart of $G$ around $e$, call it

$$c = (U, \varphi, \mathbb{Q}_p),$$

and we let $\varphi(e) = 0$.

Recall that the multiplication map

$$m : G \times G \to G$$

is locally analytic and so there exists a neighborhood...
Let \( \mathcal{U} \) restrict the chart around \( e \) now and take 
\[(Y, \varphi_{|Y}, Q^d_p)\].

We have a map:
\[
\varphi(v) \times \varphi(v) \xrightarrow{\varphi^{-1} \times \varphi^{-1}} V \times V \xrightarrow{m} U \xrightarrow{\varphi} \varphi(U)
\]

which a composition of locally analytic maps.

\[
F := \varphi \circ m \circ (\varphi^{-1} \times \varphi^{-1}) : \varphi(v) \times \varphi(v) \rightarrow \varphi(U)
\]
is locally analytic. Notice \( \varphi(v) \times \varphi(v) \subseteq Q^d_p \), \( \varphi(U) \subseteq Q^d_p \) are open sets.

**Step 2: Using the right expansion.**

By definition of locally analytic function for \( F : Q^d_p \times Q^d_p \rightarrow Q(U) \) around \( e \), we can find \( F_1, \ldots, F_d \) formal power series in \( x \) and \( y \) such that

\[
F(x, y) = (F_1(x, y), \ldots, F_d(x, y))
\]

and

\[
F_i(x, y) = \sum_{\alpha, \beta} C_{i, \alpha, \beta} x^\alpha y^\beta
\]

where \( X = (X_1, \ldots, X_d) \) and \( Y = (Y_1, \ldots, Y_d) \) and \( \alpha, \beta \) are \( d \)-tuples.
Recall that
\[ F_i : \varphi(v) \times \varphi(v) \rightarrow \Omega_p \]
is such that
\[ F_i \in \mathcal{F}_c(\Omega_p^d ; \Omega_p) \]
and the \( \Omega_i \) is the same for all \( 1 \leq i \leq d \).

We emphasize three properties now of these functions:

- **Property 1:** Recall that the sets
  \[ \mathbb{Z}_p^d \supseteq p \mathbb{Z}_p^d \supseteq p^2 \mathbb{Z}_p^d \supseteq p^3 \mathbb{Z}_p^d \supseteq \ldots \]
  form a system of compact open sets that are a neighborhood of the identity.
  Since \( \varphi(v) \) is a neighborhood of \( e \) we have that for big enough \( n \),
  \[ p^n \mathbb{Z}_p^d \subseteq \varphi(v). \]

- **Property 2:**
  Pick any \( x \) and \( y \) in \( \Omega_p^d \) and consider
  \[ \varphi^{-1}(x), \varphi^{-1}(y) \in V. \]
  Then \( \varphi^{-1}(x) \varphi^{-1}(y) \in \mathcal{U} \) and so, by definition of \( F \),
  \[ \varphi(\varphi^{-1}(x) \varphi^{-1}(y)) = (F_1(x,y), \ldots, F_d(x,y)). \]
Property 3: Since $F_i \in \mathcal{F}_E (\mathbb{Q}_p ^{2d} ; \mathbb{Q}_p)$ we have that

$$\lim_{|l| + |l| \to \infty} \| C_i \alpha, \beta \|_p \mathcal{E} = 0.$$ 

For big enough $n$ we have $p^{-n} \leq \mathcal{E}$ and so $F_i \in \mathcal{F}_{p^{-n}} (\mathbb{Q}_p ^{2d} ; \mathbb{Q}_p)$, so that we have

$$\lim_{|l| + |l| \to \infty} \| C_i \alpha, \beta \|_p p^{-n(|l| + |l|)} = 0,$$

that is,

$$\lim_{|l| + |l| \to \infty} p^{-\text{val}(C_i \alpha, \beta) - n(|l| + |l|)} = 0.$$

Equivalently, for big enough $n$,

$$\lim_{|l| + |l| \to \infty} (\text{val}_p(C_i \alpha, \beta) + n(|l| + |l|)) = \infty.$$

If we fix an $n$, then for big enough $\alpha, \beta$ we have

$$\text{val}_p(C_i \alpha, \beta) + n(|l| + |l|) \geq n$$

for every $i$, but it might fail for some $\alpha, \beta$. The point is that we can actually, increasing $n$ further, assure it doesn't.

**Lemma:** If we increase $n$, we have for all $i, \alpha, \beta$ that

$$\text{val}_p(C_i \alpha, \beta) + n(|l| + |l|) \geq n.$$
Proof of lemma:

Proving that

\[ \text{val}(C; a, b) + n \| (1 \|_1 + 1 \|_1) \| \geq n \quad \forall a, b, i \]

it's equivalent to

\[ \| F_i \|_{p-n} \leq p^{-n} \]

which is what we will prove. Denote by

\[ \varepsilon_0 := \| F_i \|_{2} \]

Then

\[ \| C; a, b \| \leq \varepsilon_0 \]

by definition of norm: and so

\[ \| C; a, b \| \leq \varepsilon_0 \varepsilon^{ -|a| - |b|} \]

Now compute: Now pick \( n \) such that

\[ p^n \geq \max \left( \frac{1}{\varepsilon}, \frac{\varepsilon_0}{\varepsilon^2} \right) \]

Compute then:

\[ \| F_i \|_{p-n} = \max \left( p^{-n}, \max \left( \| C; a, b \|_{p-n}^{(1 \|_1 + 1 \|_1)} \right) \right) \]

\[ \leq \max \left( p^{-n}, \max \varepsilon_0 \left( \frac{p^{-n}}{\varepsilon} \right)^{(1 \|_1 + 1 \|_1)} \right) \]

\[ \leq \max \left( p^{-n}, \varepsilon_0 \left( \frac{p^{-n}}{\varepsilon} \right)^2 \right) \]

\[ \leq p^{-n} \]

Comment on "proof":

while in class we realized here we put the \( p^{-n} \) because each \( F_i \) has an expansion

\[ F_i(x, y) = x + y + \sum \text{higher order terms} \]

which separates the max among those of higher terms & those with \( (1,0) \text{ & } (0,1) \) whose coefficients is 1.
Concretely, for big enough $n$, we have for all $i, \alpha, \beta$:

$$V_1 \varphi(C_{i, \alpha, \beta}) + n(1\|a1 + 1\|b) \geq n. \quad (*)$$

**Step 3:** The inverse.

We have done the previous steps for the multiplication map but we can do them as well for the inverse map, which is also locally analytic.

**Step 4:** Building subgroups.

Because $p^n \mathbb{Z}_p^d \subseteq \varphi(V)$ we know that

$$\varphi^{-1}(p^n \mathbb{Z}_p^d) \subseteq V \subseteq G$$

are compact open subsets of $G$. Notice that if $x, y \in p^n \mathbb{Z}_p^d$ then

$$\varphi^{-1}(x) \varphi^{-1}(y) = \varphi^{-1}(F_1(x, y), \ldots, F_d(x, y))$$

and precisely the fact that $(*)$ happens implies

$$F_i(x, y) \in p^n \mathbb{Z}_p^d$$

So that

$$\varphi^{-1}(x) \varphi^{-1}(y) \in \varphi^{-1}(p^n \mathbb{Z}_p^d)$$

for large enough $n$. Repeating this with the inverse we conclude that:

For large enough $n$, $\varphi^{-1}(p^n \mathbb{Z}_p^d)$ are subgroups open compact of $G$. 

Step 5: Rescaling the power series.

Pick now any of these large $n$, so that $\phi^{-1}(p^nZ_p)$ is a compact open subgroup of $G$, and define

$$\psi := \phi^\psi.$$

The multiplication map on this chart is given, in coordinates as:

$$\sum_{\alpha \in \Lambda} (p^{-n} C_{i,\alpha}) x^\alpha y^\beta.$$

in the i-th entry.

We proved in the lemma that

$$\|F\|_{p^{-n}} \leq p^{-n}.$$

That is, for all $\alpha, \beta$,

$$\|C_{i,\alpha,\beta}\|_p \leq p^{-n}.$$

That is:

$$\|p^n\|_p \|C_{i,\alpha,\beta} p^n\|_p \leq p^{-n}.$$

implying

$$\|C_{i,\alpha,\beta} p^{-n}\|_p \cdot p^{-n(1\|d\|B)} \leq 1.$$
Repeating this by iteration we have that
\[
g \rightarrow g^p,
\]
\[
(g, h) \rightarrow [g, h]
\]

have convergent power expansions in around \( e \) in \( \Psi \), converging in all of \( \mathbb{Z}_p^d \) with coefficients in \( \mathbb{Z}_p \).

We define
\[
G_1 := \Psi^{-1}(p^{-1} \mathbb{Z}_p^d).
\]