Let $G$ be any abstract group.

**Definition (p-valuation):**

A $p$-valuation $\omega$ on $G$ is a real-valued function

$$\omega : G \setminus \{1\} \rightarrow (0, \infty)$$

and, by convention, $\omega(1) = \infty$, that satisfies:

a) $\omega(g) > \frac{1}{p-1}$,

b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$

c) $\omega([g, h]) \geq \omega(g) + \omega(h)$,

c) $\omega(g^p) = \omega(g) + 1$,

for any $g, h \in G$.

Our convention for the commutator is

$$[g, h] = ghg^{-1}h^{-1}.$$

Before giving some examples we prove:

**Proposition (Properties of p-valuations):** For a $p$-valuation of $G$ we have:

i) $\omega(g^{-1}) = \omega(g)$ \quad \forall g \in G;

ii) $\omega(ghg^{-1}) = \omega(h)$ \quad \forall g, h \in G.

iii) If $\omega(g) \neq \omega(h)$, then $\omega(gh) = \min \{\omega(g), \omega(h)\}$.

**Proof:**

i) In property (b) put $h = 1$, then:
\[ \omega(g^{-1}h) = \omega(g^{-1}1) \]
\[ \geq \min \{ \omega(g), \omega(1) \} \]
\[ = \min \{ \omega(g), \infty \} \]
\[ = \omega(g). \]

Hence,
\[ \omega(g^{-1}) \geq \omega(g). \]

Changing the roles of \( g^{-1} \) and \( g \) we get
\[ \omega(g) \geq \omega(g^{-1}). \]

\[ \therefore \omega(g^{-1}) = \omega(g). \]

(ii) If \( g \) or \( h \) is 1 the proposition (ii) is clear, hence we suppose they are not 1.

We have:
\[ \omega(ghg^{-1}) = \omega(ghg^{-1}h^{-1}h) \]
\[ = \omega([g,h]h) \]
\[ \geq \min \{ \omega([g,h]^{-1}), \omega(h) \} \]
\[ = \min \{ \omega([g,h]), \omega(h) \} \]
\[ = \min \{ \omega(g) + \omega(h), \omega(h) \} \]
\[ = \omega(h) \text{ by property (c) because } \omega \text{ is non-negative.} \]

Hence \( \omega(ghg^{-1}) \geq \omega(h) \), but then:
\[ \omega(h) = \omega(g^{-1}ghg^{-1}g) \]
\[ \geq \omega(gg^{-1}) \]
\[ \text{by the first part of (ii).} \]
We conclude
\[ \omega(g h g^{-1}) = \omega(h). \]

(iii) Let's suppose \( \omega(g) > \omega(h) \) without loss of generality. If \( g=1 \) then this is clear, so suppose it is not. We have
\[
\omega(gh) = \omega((g^{-1})^{-1} h) \\
\geq \min \{ \omega(g^{-1}), \omega(h) \} \\
= \min \{ \omega(g), \omega(h) \}
\]

Now, for the other inequality,
\[
\omega(h) = \omega(g^{-1} g h) \\
\geq \min \{ \omega(g^{-1}), \omega(gh) \} \\
= \min \{ \omega(g), \omega(gh) \}
\]

but \( \omega(g) > \omega(h) \), hence this last minimum can't be \( \omega(g) \), so it is \( \omega(gh) \). That is,
\[
\min \{ \omega(g), \omega(h) \} = \omega(h) \geq \min \{ \omega(g), \omega(gh) \} = \omega(gh)
\]
as desired.

Using this we can construct for any real \( \nu > 0 \):
\[
G_{\nu} := \{ g \in G : \omega(g) \geq \nu \},
\]
\[
G_{\nu^+} := \{ g \in G : \omega(g) > \nu \}.
\]
Notice that

\[ G_{v^+} \leq G_v \]

We have:

\underline{Theorem:} For any \( v > 0 \), \( G_v \) and \( G_{v^+} \) are normal subgroups of \( G \). We also have:

- \( G_v / G_{v^+} \) is a central subgroup of \( G / G_{v^+} \).
- If we denote \( \text{gr}_v(G) := G_v / G_{v^+} \) and define

\[ \text{gr}(G) := \bigoplus_{v > 0} \text{gr}_v(G) \]

Then \( \text{gr}(G) \) is a \( \mathbb{F}_p \)-vector space.

(iii) For any \( v, v' > 0 \) the map

\[ \text{gr}_v(G) \times \text{gr}_{v'}(G) \rightarrow \text{gr}_v G_{v' + v} \]

\[ (\xi, \eta) \mapsto [\xi, \eta] := \left[ g, h \right] G_{v' + v} \]
where $g$ and $h$ are representatives of $\xi$ and $\eta$, respectively, is a well defined bi-additive map.

Notice that $[\xi, \xi] = 0$ and $[\xi, \eta] = -[\eta, \xi]$.

iv) Extending $[\cdot, \cdot]$ bilinearly to $gr(G) \times gr(G)$ we get a graded $\mathbb{F}_p$-bilinear map

$$[\cdot, \cdot] : gr(G) \times gr(G) \rightarrow gr(G)$$

with $[\xi, \xi] = 0$.

v) $[\cdot, \cdot, \cdot]$ satisfies the Jacobi identity.

Proof:

By proposition part (b) of previous proposition if $\omega(g), \omega(h) > \nu$ (or $\geq \nu$) then

$$\omega(g^{-1}h) \geq \min(\omega(g), \omega(h)) > \nu \ (\text{or } \geq \nu),$$

hence $g^{-1}h \in G_\nu$ (or $G_{\nu+}$), proving it is a subgroup.

Moreover, since

$$\omega(ghg^{-1}) = \omega(h)$$

we conclude normality. Hence we have:

$$G_\nu \triangleleft G_\delta,$$

$$G_{\nu+} \triangleleft G_\delta,$$

$$G_{\nu+} \triangleleft G_\nu.$$

(i) Since $G_{\nu+} \triangleleft G_\nu$ it makes sense the map

$$G_\nu / G_{\nu+} \rightarrow G_\nu / G_{\nu+}.$$
defined by "inclusion"

\[ g Gv_t \rightarrow g Gv_t \]

(in Gv)

and it is a homomorphism. We just note it is injective since

persistence to Gv_t doesn't change.

Pick any \( h Gv_t \) in \( G/Gv_t \) and \( g Gv_t \) with \( g \in Gv \).

Then

\[(g Gv_t)(h Gv_t)(g^{-1} Gv_t)(h^{-1} Gv_t)\]

\[= (ghg^{-1}h^{-1}) Gv_t\]

\[= [g, h] Gv_t,\]

but

\[\omega([g, h]) \geq \omega(g) + \omega(h)\]

\[> \omega(g)\]

\[> \gamma,\]

so that \( [g, h] \in Gv_t \). This proves \( Gv/Gv_t \) is central in \( G/Gv_t \). Restricting to itself proves also the commutativity.

(ii) Because of commutativity we know \( gr(G) \) is also a commutative group and we only need to define scalar multiplication.

Over \( \mathbb{Z} \) is clear the definition

\[n \cdot x := \underbrace{x + \ldots + x}_{n \text{-times}}\]
and we need to prove
\[ p \cdot x = 0 \quad \forall x \in \text{gr}(G) \]
We prove this in homogeneity, that is, take \( x \in \text{gr}_u(G) \) for some \( u > 0 \). Say
\[ x = g \cdot G_{u^+}, \quad g \in G_{u^+} \]
Then
\[ p \cdot x = \cdots p x = (g \cdot G_{u^+}) \cdots (g \cdot G_{u^+}) = g^p \cdot G_{u^+} \]
and \( \omega(g^p) = \omega(g) + 1 > \omega(g) \geq u \). That is, \( g^p \in G_{u^+} \), i.e. \( p \cdot x = 0 \).

The product descends and we have a scalar multiplication by \( \mathbb{F}_p \).

(iii) Let us prove that it is well defined. Say
\[ x = g, \quad G_{u^+} = g_2 \cdot G_{u^+}, \quad g, g_2 \in G_{u^+} \]
\[ y = h, \quad G_{v^+} = h_2 \cdot G_{v^+}, \quad h, h_2 \in G_{v^+} \]

We need
\[ [g_1, h_1] \cdot g_{(u+v)^+} = [g_2, h_2] \cdot g_{(u+v)^+}. \] \( (*) \)

Firstly, notice that for \( i = 1, 2 \):
\[ \omega([g_i, h_i]) = \omega(g_i) + \omega(h_i) \geq u + v', \]
proving \( [g_i, h_i] \in G_{u^+} \) so that the cost makes sense.
For any real number \( r > 0 \) we already proved that
\[
g_r(G) < g/G_r^+
\]
if a central subgroup, hence if \( x \in G_r \) and \( y \in G \) and we look at cosets we have:

\[
(y \times y^{-1}) G_r^+ = x G_r^+
\]

Hence representatives in \( G_r \) can be conjugated without affecting class.

Suppose
\[
\varphi E_{G_r^+}, \quad \varphi = g G_r^+, \quad \eta = h_1 G_r^+ = h_2 G_r^+.
\]

we have
\[
\omega([g, h_i]) \geq \omega(g) + \omega(h_i) > v + v',
\]
so that
\[
[g, h_i] \in G(v + v')^+.
\]

and taking coset makes sense. We will prove
\[
[g, h_i] G(v + v')^+ = [g, h_2] G(v + v')^+.
\]

write \( h_2, h_1, \varphi = k \in G(v + v')^+ \in G \). Then in \( G/G(v + v')^+ \) we are conjugate to have:
\[
[g, h_i] G(v + v')^+ = g h_i g^{-1} h_i^{-1} G(v + v')^+.
\]
\[ g(h_2 k) g^{-1} (h_2 k)^{-1} G_{(v+v)^+} \]
\[ = g h_2 k g^{-1} k^{-1} h_2^{-1} G_{(v+v)^+} \]
\[ = g h_2 g^{-1} g k g^{-1} k^{-1} h_2^{-1} G_{(v+v)^+} \]
\[ = g h_2 g^{-1} [g, k] h_2^{-1} G_{(v+v)^+} \]
\[ = g h_2 g^{-1} h_2^{-1} h_2 [g, k] h_2^{-1} G_{(v+v)^+} \]
\[ = [g, h_2] \cdot h_2 [g, k] h_2^{-1} G_{(v+v)^+} \]
\[ = [g, h_2] G_{(v+v)^+} \cdot h_2 [g, k] h_2^{-1} G_{(v+v)^+} \]

A priori this splitting happens in \( G / G_{(v+v)^+} \), but

\[ w([g, k]) \geq w(g) + w(k) > v + v' \]

where inequality is strict since \( k \in G_{v^4} \), and so

\[ [g, k] G_{(v+v)^+} = G_{(v+v)^+} \]

and we can conjugate by what we said before, hence

conjugating by \( h_2 \) we still get \( G_{(v+v)^+} \).

\[ [g, h] G_{(v+v)} = [g, h_2] G_{(v+v)^+} \]

Analogously if we hold \([g_1, h] \) and \([g_2, h] \). Hence

\[ [g_1, h] G_{(v+v)^+} = [g_1, h_2] G_{(v+v)^+} = [g_2, h_2] G_{(v+v)^+} \]
The philosophy of this is, basically, do computations in $G/G_{r_+}$ and return to $G/G_{r_+}$ since it is central in it.

Additivity is:

$$[g, G_{v_+} + g_2 G_{v_+}, h G_{v_+}] = [g, G_{v_+}, h G_{v_+}] + [g_2 G_{v_+}, h G_{v_+}]$$

that is:

$$[g, g_2, h] G_{(v^+ v)_+} = [g, h] [g_2, h] G_{(v^+ v)_+}$$

which now follows from the equality:

$$[gh, k] = g [h, k] g^{-1} [g, k]$$

since then:

$$[g, g_2, h] G_{(v^+ v)_+} = g [g_2, h] g^{-1} [g, h] G_{(v^+ v)_+}$$

$$= g_1 [g_2, h] g_1^{-1} G_{(v^+ v)_+} + [g_1, h] G_{(v^+ v)_+}$$

$$= [g_2, h] G_{(v^+ v)_+} + [g_1, h] G_{(v^+ v)_+}$$

$$= [g_1, h] G_{(v^+ v)_+} G_{(v^+ v)_+}$$

since we can conjugate.

by commutativity.
(iv) The follow immediately from (iii).
(v) The follow in the same way from

\[ [g, h], h k h^{-1} ][h, k], k g k^{-1}][k, g], g h g^{-1} = 1. \]

Because of this proposition \( \text{gr}(G) \) is a Lie Algebra over \( \mathbb{F}_p \). For the next proposition let us suppose for a moment \( p=2 \). Then:

\[ h^{-2} g^{-2} (gh)^2 = [h^{-2}, g^{-1}][h^{-1}, g^{-1}]^{-1} \]

Indeed we have:

\[ [h^{-2}, g^{-1}][h^{-1}, g^{-1}]^{-1} = h^{-2} g^{-1} h^2 g (h^{-1} g^{-1} h g)^{-1} \]
\[ = h^{-2} g^{-1} h^2 g g^{-1} h^{-1} g h \]
\[ = h^{-2} g^{-1} h^2 h^{-1} g h \]
\[ = h^{-2} g^{-1} h g h \]
\[ = h^{-2} g^{-2} g h g h \]
\[ = h^{-2} g^{-2} (gh)^2. \]

Notice then that

\[ \omega(h^{-2} g^{-2} (gh)^2) = \omega([h^{-2}, g^{-1}][h^{-1}, g^{-1}]^{-1}) \]
\[ \geq \min \{ \omega([h^{-1}, g^{-1}]), \omega([h^{-2}, g^{-1}]) \} \]

But \( \omega([h^{-1}, g^{-1}]) \geq \omega(h^{-1}) + \omega(g^{-1}) = \omega(h) + \omega(g) \).
and
\[
\omega([h^{-2}, g^{-1}]) \geq \omega(h^{-2}) + \omega(g^{-1})
\]
\[
= \omega(h^{-1}) + \omega(g^{-1}) + 1
\]
\[
\geq \max \{ \omega(h^{-1}), \omega(g^{-1}) \} + 1
\]
\[
= \max \{ \omega(h), \omega(g) \} + 1.
\]

We conclude:
\[
\begin{align*}
\omega(h^{-2}g^{-2}(gh)^2) &\geq \min \{ \omega([h^{-1}, g^{-1}]), \omega([h^{-2}, g^{-1}]) \} \\
\omega([h^{-2}, g^{-1}]) &> \max \{ \omega(h), \omega(g) \} + 1 \\
\omega([h^{-1}, g^{-1}]) &\geq \omega(h) + \omega(g)
\end{align*}
\]

but property (a), that we haven't used so far implies
\[
\omega(h) > 1, \quad \omega(g) > 1 \quad (p=2).
\]

Hence (*) becomes
\[
\omega([h^{-1}, g^{-1}]) \geq \max \{ \omega(h), \omega(g) \} + 1.
\]

In conclusion:
\[
\boxed{\omega(h^{-2}g^{-2}(gh)^2) > \max \{ \omega(h), \omega(g) \} + 1.}
\]

For general \(p\) we have:

**Proposition:** For any two elements \(g, h \in G\) we have

(i) \(\omega(h^{-p}g^{-p}(gh)^p) > \max \{ \omega(g), \omega(h) \} + 1\)

(ii) \(\omega(g^{-p}h^n) = \omega(g^{-1}h) + n\) for \(n \geq 1\).

For \(h^n(\text{or } j^p, k^n) \geq \omega(g) + \omega(h) + 1.\)
Comments on proof(s):

(i) It is done in the same way, but the rewriting requires to write

\[ e^{h-p} g^{-p} (gh)^p = A \cdot B \]

where \( A \) can be written as a product of brackets with \( p \)-th powers, whereas \( B \) can be written as iterated brackets exactly \( p-1 \) of them. When computing this inequality \( \omega(x) \geq \frac{1}{p-1} \) is used to exchange one \( \omega(g) \) and cancel a factor of \( p-1 \) that appears: suppose \( X_1, \ldots, X_p \) are \( g, h, g^{-1}, h^{-1} \) and suppose \( \omega(g) \geq \omega(h) \). Then:

\[
\omega \left( [X_1, [X_2, \ldots, [X_{p-1}, X_p] \ldots ] ] \right) \\
\geq \omega(X_1) + \omega([X_2, \ldots ] ) \\
\geq \omega(X_1) + \ldots + \omega(X_p)
\]

If all the \( X_i = h \) or \( h^{-1} \) then LHS is \( \infty \) since the bracket is \( \infty \), hence there is at least one \( g \) and the rest are \( \geq \omega(h) \).

Hence we have:

\[
\omega \left( [X_1, [X_2, \ldots, [X_{p-1}, X_p] \ldots ] ] \right) \geq \omega(g) + (p-1) \omega(h) \\
> \omega(g) + 1 \\
= \max \{ \omega(g), \omega(h) \} + 1.
\]

In this way one gets \( \omega(A), \omega(B) \geq \max \{ \omega(g), \omega(h) \} + 1 \).
Now pick \( \alpha \). \( \nu > 0 \). We define a map

\[
\begin{align*}
gr_{\nu}(G) & \longrightarrow gr_{\omega(1)+}(G) \\
g \cdot G_{\nu+} & \longrightarrow g^\nu G_{\omega(1)+}
\end{align*}
\]

We need to prove that this is well defined. Suppose then

\[
g = g_0 h, \quad g_0, g \in G_{\nu}, h \in G_{\nu+}.
\]

Then:

\[
g^\nu G_{\omega(1)+} = (g_0 h)^\nu G_{\omega(1)+} \quad \text{since } \omega(h)^\nu g_0^\nu (g_0 h)^\nu \\
= g_0^\nu h^\nu G_{\omega(1)+} \quad \text{since } \omega(h)^\nu = \omega(h) + 1 > \nu + 1 \\
= g_0^\nu G_{\omega(1)+} \cdot h^\nu G_{\omega(1)+} \quad \text{since } h^\nu \in G_{\omega(1)+}.
\]

\[
\therefore It is well defined.
\]
Varying \( \gamma \) we can then define:

\[
P: \text{gr}(G) \longrightarrow \text{gr}(G),
\]

by defining it in homogeneous elements.

**Theorem:**

i) \( P \) is an \( \mathbb{F}_p \)-linear mapping of degree 1.

ii) \( \mathbb{F}_p [P] \) is torsion free.

iii) \( \text{gr}(G) \) with the bracket defined as before is a Lie Algebra over \( \mathbb{F}_p [P] \).

**Proof:**

i) That it is linear follows from

\[
P(g Gv_t + h Gv_t) = P(gh Gv_t)
\]

\[
= (gh)^P Gv_t
\]

\[
= g^P h^P Gv_t
\]

\[
= g^P Gv_t + h^P Gv_t.
\]

\[
= P(g Gv_t) + P(h Gv_t).
\]

Of course this implies: for \( 0 \leq i < p \) that

\[
P(i \cdot g Gv_t) = P( g Gv_t + \ldots + g Gv_t)
\]

\[
= P(g Gv_t) + \ldots + P(g Gv_t)
\]

\[
= i \cdot P(g Gv_t),
\]

proving linearity.
Suppose \( q(P) = a_n P^n + a_{n-1} P^{n-1} + \ldots + a_0 P^0 \rangle \) such that \( \xi \in \text{gr}(G) \) is such that
\[
q(P) \cdot \xi = 0.
\]
Expanding \( \xi \) in homogeneous elements, all non-zero,
\[
\xi = \sum_{i=1}^{t} \xi_{v_i}
\]
with \( v_1 < v_2 < \ldots < v_t \). We then have:
\[
0 = q(P) \cdot \xi = \left( \sum_{j=0}^{n} a_j P^j \right) \cdot \left( \sum_{i=1}^{t} \xi_{v_i} \right)
\]
\[
= \sum_{j=0}^{n} \sum_{i=1}^{t} a_j P^j (\xi_{v_i})
\]
\[
= a_n P^n \cdot \xi_{v_t} + \text{(the rest)}
\]
notice that \( P^n \) is of degree \( n \), hence
\[
P^n \cdot \xi_{v_t} \in \text{gr}_{v_t+n} (G)
\]
whereas all other terms have \( j < n \) or \( v_i < v_t \) so that
\[
P^j (\xi_{v_i}) \in \text{gr}_{v_t+j} (G), \quad v_i + j < v_t + n.
\]
:. \( P^n (\xi_{v_t}) \) is the only one in \( \text{gr}_{v_t+n} (G) \), implying
\[
a_n \cdot P^n (\xi_{v_t}) = 0.
\]
Since \( a_n \neq 0 \) we have \( P^n (\xi_{v_t}) = 0 \). Notice that by definition\( P \) is injective. Hence \( P^n \) too and so \( \xi_{v_t} = 0 \). This is a contradiction. :: \( \text{gr}(G) \) is torsion free.
We just need to prove
\[ [P_\xi, \eta] = P([\xi, \eta]) \]
The last property of previous proposition says
\[ \omega([g, h] \cdot [g^p, h]) > \omega(g) + \omega(h) + 1 \]
which means that if
\[ \xi = g G_{s+t}, \quad g \in G_s, \]
\[ \eta = h G_{s+t}, \quad h \in G_s \]
then
\[ [P_\xi, \eta] = [g^p G_{s+t}, h G_{s+t}] = [g^p, h] G_{s+t+t} \]
and
\[ P([\xi, \eta]) = P([g G_{s+t}, h G_{s+t}]) = P([g, h] G_{s+t+t}) = [g, h]^p G_{s+t+t} \]
are equivalent! This proves what we want.

**Definition:** Let \( G \) be a group with a \( p \)-valuation \( \omega \). We say \( (G, \omega) \) is of **finite rank** if \( \text{gr}(G) \) is finitely generated as an \( F_p [P] \)-module.

Notice that \( F_p [P] \) is isomorphic to \( F_p [X] \) since \( P \) cannot satisfy any relation since \( \text{gr}(G) \) is torsion-free. Hence \( F_p [P] \) is a principal ideal domain.

If \( (G, \omega) \) is of finite rank, then it is actually free over \( F_p [P] \).
Definition: We define the rank of \((G, \omega)\) to be

\[
\text{rank}(G, \omega) := \text{rank}_{\Phi_{2P}}(\text{gr } G).
\]