The tangent plane.

Let $M$ be a manifold. We consider pairs $(c, v)$ where $c = (U, \varphi, \mathbb{R}^n)$ is a chart of $M$ and $v \in k^n$. Fix $a \in M$.

**Definition:** Two pairs $(c, v)$ and $(c', v')$ are equivalent if

$$D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v) = v', \quad \varphi'(a) = \varphi(a).$$

Recall that different charts in $M$ are equivalent since we are fixing a maximal atlas, so we have:

Recall we proved that $\varphi' \circ \varphi^{-1}$ is strictly diffeomorphic. Since it is locally analytic and that compatible charts have the same dimension. Hence $D_{\varphi(a)}(\varphi' \circ \varphi^{-1}) : k^n \rightarrow k^m$ indeed exists and the condition we are asking makes sense.
This is clearly an equivalence relation and hence we define:

**Definition:** A tangent vector of $M$ at $a \in M$ is an equivalence relation class $[c, v]$ of pairs $(c, v)$ as above. The set of all tangent vectors is called the tangent plane at $a$.

**Proposition:** Let $c = (U, \varphi, k^m)$ and $c' = (U', \varphi', k^m)$ be two charts for $M$ around $a$. Then

1. The map

   $\Theta_c : k^m \longrightarrow T_a(M)$

   $v \longmapsto [c, v]$ is bijective.

2. $\Theta_{c'} \circ \Theta_c : k^m \longrightarrow k^m$ is a $k$-linear isomorphism.

**Proof:**

1. To prove injectivity: suppose $[c, v] = [c, v']$. Then we have

   $D_{\varphi(a)} (\varphi' \circ \varphi^{-1})(v) = v'$

   but we can put $\varphi' = \varphi$ since we are using the same chart $c$. Hence:

   $D_{\varphi(a)} (\text{id})(v) = v'$

   that is, $v = v'$.

To prove surjectivity: Suppose $[c, v]$ is a tangent vector. We need to find $v'$ such that $[c, v] = [c, v']$. That is,

$D_{\varphi(a)} (\varphi' \circ \varphi^{-1})(v) = v'$
But \( \varphi' \circ \varphi^{-1} \) is an homeomorphism, hence its derivative is to, hence we can pick
\[
v = D_{\varphi'(a)} (\varphi' \circ \varphi^{-1})^{-1} (v')
\]
and this makes the job.

(ii) We need to understand the inverse of \( \Theta_c \) to get this part. If we have
\[
[c, v] = \Theta_c (v) = [c', v']
\]
then we have
\[
v = D_{\varphi'(a)} (\varphi' \circ \varphi^{-1}) (v')
\]
That is:
\[
\Theta_c^{-1} ([c', v']) = D_{\varphi'(a)} (\varphi' \circ \varphi^{-1}) (v')
\]
Hence:
\[
\Theta_c^{-1} \circ \Theta_c (v) = \Theta_c^{-1} ([c, v])
= D_{\varphi'(a)} (\varphi' \circ \varphi^{-1}) (v')
\]
That is:
\[
\Theta_c^{-1} \circ \Theta_c = D_{\varphi'(a)} (\varphi' \circ \varphi^{-1})
\]
which we know is a \( k \)-linear isomorphism (this includes continuity)!

Hence, given this proposition, we can equip \( Ta(M) \) with a topological vector space structure by claiming \( \Theta_c \) is a topological isomorphism, and by (ii) this...
is chart independent.

Notice this definition implies that if \( M \) has dimension \( n \) iff \( T_a(M) \) has dimension \( n \) for all \( a \).

Having defined \( T_a(M) \) for each \( a \in M \), we define, as a set,

\[
T(M) = \bigsqcup_{a \in M} T_a(M).
\]

and define

\[
P_M: T_0(M) \longrightarrow M
\]

\[
t \longmapsto a
\]

if \( t \in T_a(M) \). We call this the \textbf{projection map}.

Pick a chart of \( M \), say \( \mathcal{C} = (U, \phi, k^m) \) and consider

\[
U \times k^m \xrightarrow{\tau_c} P_M^{-1}(U) (= \bigsqcup_{a \in U} T_a(M))
\]

\[
(u, v) \longmapsto [v, c] \in T_u(M).
\]

Notice that

\[
\tau_c(u, v) = [v, c] = \theta_c(u), \text{ seen at } u.
\]

Hence it is a bijection since the \( \theta_c \) were. We can then compose:

\[
P_M^{-1}(U) \xrightarrow{\tau_c^{-1}} U \times k^m \xrightarrow{\phi \times \text{id}} k^m \times k^m = k^{2m}
\]
With this we have constructed triples:

\[ (p^{-1}_M(u), \varphi, k^m) \]

as \( c \) varies. (We want these to be charts for \( T(M) \)).

**Definition:** A subset \( X \subseteq T(M) \) is called open if

\[ T^{-1}_c (X \cap p^{-1}_M(u)) \]

is open in \( U \times k^m \) for any chart \( c = (U, \varphi, k^m) \) of \( M \).

These definitions of openness defines a topology on \( T(M) \).

We now proceed to prove it is a manifold with the previous charts.

**Theorem:** i) The map \( T_c : U \times k^m \to p^{-1}_M(u) \) is a homeomorphism with respect to the subspace topology.

ii) \( p_M \) is continuous.

iii) \( T(M) \) is Hausdorff.

iv) The constructed charts are compatible.

**Proof:**

Let's pick two charts \( c = (U, \varphi, k^m) \) and \( c' = (V, \psi, k^m) \) with \( U \cap V \neq \emptyset \). We have:

\[ P^{-1}_M(U) \cap P^{-1}_M(V) \]

\[ (U \cap V) \times k^m \]

\[ T_c \]

\[ P^{-1}_M(U \cap V) \]

\[ T_c \]

\[ (U \cap V) \times k^m \]

\[ P_M \]

\[ U \cap V \]

\[ (x, v) \to [c', v] \in T_x(M) \to (x, D_{p(x)}(\varphi \circ \varphi)(v)) \cdot (x, v) \]
If we go further down to coordinates \( \varphi(U \cap V) \times \psi(U \cap V) \) we get:

\[
(x, v) \mapsto (\varphi \circ \varphi^{-1}(x), D_x (\varphi \circ \varphi^{-1})(v)).
\]

By hypothesis of compatibility the first entry is locally analytic. Now we need to understand the map:

\[
(x, v) \mapsto D_x (\varphi \circ \varphi^{-1})(v).
\]

This can be further decomposed:

\[
(x, v) \mapsto (D_x (\varphi \circ \varphi^{-1}), v) \mapsto D_x (\varphi \circ \varphi^{-1})(v)
\]

\[
k^{2m} \mapsto \mathcal{L}(k^{m}, k^{m}) \times k^{m} \mapsto k^{m}
\]

In here we use the fact:

Fact [See Pap 6.1]: Suppose \( f: U \to V \) is locally analytic, then \( f \) is strictly diff. and \( x \mapsto D_x f \) is locally analytic in \( \mathcal{C}^\omega(U, k^k, V) \).

Hence the previous map is composition of locally analytic ones (in particular, continuous and by changing rules, a homeomorphism.)

This actually proves (cv).

But we will use it for (c) as well. Pick \( Y \subseteq U \times k^m \) be open. We want to prove \( T_{(c)}(Y) \) is open in \( P^{-1}(U) \subseteq T(M) \). For that, using the definition, we need

\[
T_{(c)}^{-1}(T_{(c)}(Y) \cap P^{-1}(V)) \subseteq V \times k^m
\]

is open for any other chart \( \tilde{c} = (V, \tilde{\psi}, k^n) \). WLOG we have \( V \cap U \neq \emptyset \) so that \( n = m \).
we have:

$$T_c^{-1}(T_c(Y) \cap p^*_M(v)) = T_c^{-1}(T_c(Y) \cap p^*_M(v) \cap p^*_M(u))$$

$$= T_c^{-1}(T_c(Y) \cap p^*_M(uv))$$

$$= T_c^{-1}(T_c(Y \cap ((uv) \times k^n)))$$

and this is open since $Y$ is.

This proves (i).

To prove (ii): Notice that $p_M^{-1}(u) = T_c(U \times k^n)$ which is open.

To prove (iii): If $s \neq t$ are such that $p_M(s) \neq p_M(t)$, we find disjoint sets $U, V$ in $M$ with $p_M(s) \in U, p_M(t) \in V$ and so $p_M^{-1}(U) \cap p_M^{-1}(V) = \emptyset$

If $a = p_M(s) = p_M(t)$ and take a chart $U$ around $a$. Since $p_M^{-1}(U) \equiv U \times k^n$ and this is Hausdorff separate them there.

$\therefore$ It is Hausdorff.

Thanks to this proposition we can put a maximal atlas to $T(M)$.

**Definition:** $T(M)$ equipped with the previous topology and the maximal atlas corresponding to the defined chart is called $\mathfrak{T}$ the tangent bundle of $M$.

Notice that if $\dim M = m$, then $\dim T(M) = 2m$. 

Given a map \( f : M \rightarrow N \) between manifolds, we know it is locally analytic if "locally" it is locally analytic. That is, for all \( a \in M \exists \) a chart \( U \ni a \ni V \ni f(a) \) with \( f(U) \subseteq V \) such that

\[
y \circ f \circ \varphi^{-1} : \varphi(U) \rightarrow \varphi(V)
\]

is locally analytic.

Then taking its derivative at the point \( x \in \varphi(U) \) we have a map:

\[
D_x (y \circ f \circ \varphi^{-1}) : k^m \rightarrow k^n
\]

and composing with \( \Theta_c \) we get a map

\[
\begin{array}{ccc}
T_{\varphi(x)} M & \xrightarrow{T_{\varphi(x)}(f)} & T_{\varphi(f(x))} N \\
\downarrow \Theta_c & & \downarrow \Theta^N_c \\
k^m & \overset{D_x (y \circ f \circ \varphi^{-1})}{\longrightarrow} & k^n
\end{array}
\]

This does not depend on the charts, \( c \) or \( \varphi \). We can further define then a map

\[
T(f) : TM \rightarrow TN \\
a \mapsto T_{\varphi(a)}(f)(a)
\]

we have the following

**Proposition:** Let \( M \) be a manifold.

i) The projection map \( \pi_M \) is locally analytic.
ii) For any locally analytic map $f: M \rightarrow N$ the map $Tf: TM \rightarrow TN$ is locally analytic.

iii) We have the chain rule

$$T(f \circ g) = Tf \circ Tg.$$ 

iv) For any two manifolds $M$ and $N$ we have

$$T(M \times N) \cong TM \times TN$$

as manifolds.

The reason we are interested in $TM$ is to define the following:

**Definition:** Let $M$ be a manifold and $U \subseteq M$ an open set. A vector field $\xi$ of $M$ on $U$ is a locally analytic map

$$\xi: U \rightarrow T(N)$$

satisfying $p_M \circ \xi = id_U$. The set of all vector fields on $U$ is denoted by $\mathfrak{X}(U, TM)$.

Next class we are going to study the structure of vector fields in detail. For now we are going to move on.

**Definition:** A Lie group $G$ is a manifold which also carries the structure of a group such that the multiplication map $m$ is locally analytic.
The basic properties are:

**Properties:**

1. For any \( h \in G \), we have
   \[
   l_h : G \to G, \quad r_h : G \to G
   \]
   \[
   g \mapsto hg, \quad g \mapsto gh
   \]
   are locally analytic isomorphisms.

2. For any two elements \( g, h \in G \) the map
   \[
   T_g \left( l_{h g^{-1}} \right) : T_g(G) \to T_h(G)
   \]
   is a \( k \)-linear isomorphism; in particular
   \[
   T_e \left( l_g \right) : T_e(G) \to T_g(G)
   \]
   is an isomorphism for any \( g \in G \).

3. Any Lie Group is \( n \)-dimensional with \( n = \dim_e G \).

4. The diagram
   \[
   T_{(g,h)}(G \times G) \xrightarrow{T(m)} T_{gh}(G)
   \]
   \[
   T_{(p_1)} \times T_{(p_2)} \xrightarrow{T(g) \times T(h)} T_g(G) \times T_h(G)
   \]
   \[
   T_g(r_h) + T_h(l_g)
   \]
   is commutative. In particular
   \[
   T_{(e,e)}(G \times G) \xrightarrow{T(m)} T_e(G)
   \]
   \[
   T_e(G) \times T_e(G)
   \]
   is commutative.
Proof:
(i), (ii) and (iii) are straightforward.

We do (iv):

Define maps

\[ \tilde{z}_h : G \rightarrow G \times G \]
\[ z \rightarrow (x, h) \]

\[ j_g : G \rightarrow G \times G \]
\[ g \rightarrow (g, x) \]

These are locally analytic maps which satisfy

\[ \Pr_1 \circ \tilde{z}_h = \text{id}_G, \quad \Pr_2 \circ \tilde{z}_h = h \]

\[ \Pr_2 \circ j_g = \text{id}_G, \quad \Pr_1 \circ j_g = g \]

Hence taking tangent map:

\[ T \left( \Pr_1 \right) \circ T(\tilde{z}_h) = \text{id}_{T(G)} \quad \left( \ast \right) \]

\[ T \left( \Pr_2 \right) \circ T(\tilde{z}_h) = 0 \]

We know that the isomorphism

\[ T(G_1 \times G_2) = T(G_1) \times T(G_2) \]

is the one induced by the projections. Hence

\[ G \xrightarrow{\tilde{z}_h} G \times G \]

\[ \xrightarrow{\text{d} \circ \text{Tangent}} \]

\[ T_G \xrightarrow{T(\tilde{z}_h)} T(G \times G) \xrightarrow{T(\Pr_1) \times T(\Pr_2)} T(G) \times T(G_1) \]

is exactly \( \tilde{t} \mapsto (t, 0) \) by \( \left( \ast \right) \).
In analogous way
\[ T_h \rightarrow T(G \times G) \rightarrow T_g \times T_h. \]

Hence if \((t_1, t_2) \in T_g \times T_h\) then:
\[
(T_{P_1} \times T_{P_2})^{-1}(t_1, t_2) = (T_{P_1} \times T_{P_2})^{-1}(Q \circ T_g(i_h)(t_1) + Q \circ T_h(j_g)(t_2)) = T_g(i_h)(t_1) + T_h(j_g)(t_2).
\]

So we find the explicit form of the inverse. With this lets compute:
\[
T_{(g, h)}(m) \circ (T_{g, h}(P_{i, h}) \times T_{g, h})^{-1}
\]
\[
= T_{(g, h)}(m) \circ (T_g(i_h) + T_h(j_g))
\]
\[
= T_g(mo_i_h) + T_h(mo_j_g)
\]
\[
= T_g(c_h) + T_h(l_g).
\]

as desired. In particular, multiplying at identity correspond to adding.

**Proposition:**

The inverse is locally analytic.

**Proof:**

Consider the map
\[
\mu : G \times G \rightarrow G \times G
\]
\[(x, y) \mapsto (xy, y).\]
It is a bijection and it is locally analytic. Look at the diagram:

\[
T_{(g,n)}(G \times G) \xrightarrow{T_{g,n}(h,y)} T_{(gh,n)}(G \times G)
\]

\[
\xrightarrow{(T_{g,n}(m), \ldots)} \xrightarrow{\simeq} \xrightarrow{\simeq}
\]

\[
T_g(G) \times T_n(G) \xrightarrow{(l_1, t_2)} T_{gm}(G) \times T_{n}(G) \xrightarrow{(T_g(r_n)(l_1) + T_h(l_n)l_2, t_2)}
\]

This is commutative by (iv). Let's see the lower row and assume \((t_1, t_2)\) is in the kernel. Hence \(t_2 = 0\) and so

\[
T_g(r_n)(l_1) = 0
\]

This is an isomorphism.

so that \(t_1 = 0\).

\[\therefore\] This lower row is injective, hence so is the map \(T_{g,n}(\mu)\). Since all adjoint vector spaces are of finite dimension we conclude \(T_{g,n}(\mu)\) is a bijection.

We conclude by the inverse function theorem that the inverse is a locally analytic map.

Now write the inverse as:

\[
G \xrightarrow{\Omega} G \times G \xrightarrow{\mu^{-1}} G \times G \xrightarrow{Pr_1} G.
\]
This left pod

\[ \mathbf{C} \times \mathbf{D} \]

**Corollary:**

For every \( n \in \mathbb{Z} \) the map \( \mathbf{g} \) coincides with multiplication by \( n \).
(ii) $n = 0$ is trivial.
$n < 0$ follows by (i) and core $n > 0$ by chain rule.

Now notice the diagram:

```
\begin{align*}
  & \text{Te}(G) \\
  \downarrow & \phantom{\text{Te}(G)} \\
  & \text{Te}(G) \\
  \downarrow & \phantom{\text{Te}(G)} \\
  & \text{Te}(G) \\
\end{align*}
```

This is commutative and gives what we want.